Numerical Mathematics 2, test 1
December 8, 2022

Duration: 1 hour and 30 minutes

In front of the questions one finds the points. The sum of the points plus 1 gives the end mark of the test.

1. Consider the linear problem

$$\begin{bmatrix} 10^{-20} & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

   We consider this problem on a computer which uses a unit round-off error of 1e-16.

   (a) [0.2]Determine the condition number of the matrix in the infinity norm.

   (b) [1.0] Give the solution if we use Gaussian elimination without pivoting.

   (c) [1.0] Give the solution if we use Gaussian elimination with partial pivoting and scaling.

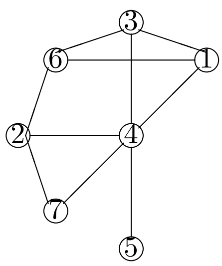
   (d) [0.3] Which of the two approaches gives the correct approximate solution and why?

   (e) Let $A = LU$.

      i. [0.4] Proof that $\kappa(A) \leq \kappa(L)\kappa(U)$ where $\kappa(A)$ denotes the condition number of A.

      ii. [0.5] Specify the L's and U's associated to parts b and c and determine their respective condition numbers. For which case $\kappa(L)\kappa(U)$ gets closest to $\kappa(A)$. Why is it relevant that $\kappa(L)\kappa(U) \approx \kappa(A)$?

2. Consider the graph of a symmetric matrix A depicted below.



   (a) [0.4] Make a sketch of the associated matrix (with * if element in matrix is non-zero)

   (b) [1.2] Determine the minimum degree ordering and sketch the matrix after reordering. Also motivate your choices.

   (c) [0.8] Sketch the L factor of Gaussian elimination without pivoting of the matrix in the previous part.

Test continues at other side

3. Consider a system $Ax = b$ of the form

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

(a) [0.2] Show that this linear problem does not have a solution.

(b) [0.8] Show that the singular values of *any* symmetric A are equal to the absolute values of the eigenvalues of A.

(c) [1.2] Compute the SVD factorization of the matrix of our problem.

(d) [1.0] Compute the pseudo inverse of A using the SVD. and use it to compute an approximate solution of the above system $Ax = b$.